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BOUNDARY-VALUE PROBLEM OF DRAINAGE IN A FRESH GROUNDWATER FRINGE ABOVE SALINE GROUNDWATER

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A multiparameter boundary-value problem of fresh infiltration water seepage in a drained fringe above quiescent saline water is solved in the direct statement and studied in detail.

Key words: *fresh water, infiltration, mapping parameters, critical drainage regime.*

In a previous study [1], we solved the problem of steady flow to horizontal drain pipes in a fresh ground water fringe above saline water with uniform infiltration onto the free surface of the fringe and studied the limiting drainage regimes for each of the two free boundaries of the fringe: the free surface and the interface. These results are used in the present paper to perform a complete analysis of the indicated filtration process in the direct statement of a boundary-value problem.

1. Formulation and Solution of the Problem. At a certain time, infiltration at a rate ϵ distributed uniformly over the area occurs onto a soil containing a fresh water layer of thickness M_0 above saline water. Simultaneously, equidistant horizontal pipe drains modeled by point sinks with identical filtration rates compensating for the infiltration are actuated at a height T_0 in the layer above the saline water surface. We assume that the saline water is insulated from external sources and sinks, and, hence, its initial volume does not change. The volume of the fresh water layer is also kept constant; it becomes a so-called fringe. The periodicity of the fringe flow due to the adopted assumptions allows us to confine ourselves to analyzing the process in one of the half-periods (Fig. 1).

The boundary-value problem corresponding to the above flow pattern consists of finding the complex flow potential $\omega = \varphi + i\,\psi$ (φ is the filtration velocity potential and ψ is the stream function) normalized by the quantity $æL$ ($æ$ is the soil permeability and L is half the distance between neighboring drains) as a function of the complex coordinate $z = x + i\,y$ which is analytic in the flow region and normalized by L under the boundary conditions

$$CD\colon\quad x = 0,\quad \psi = 0;\qquad ED\colon\quad x = 0,\quad \psi = \epsilon;\qquad AG\colon\quad x = 1,\quad \psi = \epsilon;$$

$$AC\colon\quad \varphi + y = 0,\quad \psi - \epsilon x = 0;\qquad EG\colon\quad \varphi - \rho y = \text{const},\quad \psi = \epsilon \tag{1}$$

$$[\rho = (\rho_2 - \rho_1)/\rho_1].$$

The first condition on the segment EG is based on the assumptions that the saline water is at rest during fresh water seepage and that the pressure is continuous in passage through the interface between the fresh and saline water [2], whose densities are equal to ρ_1 and ρ_2, respectively.

The formulated problem is solved using the analytic theory of linear differential equations, whose fundamentals for two-dimensional steady filtration problems were developed by Polubarinova-Kochina [2]. The goal of the method is to find the functions $\Omega = d\omega/d\zeta$ and $Z = dz/d\zeta$ defined in the half-plane $\operatorname{Im}\zeta \geqslant 0$ of the auxiliary complex variable $\zeta = \xi + i\,\eta$ (Fig. 2). Since in the problem considered, as in other plane filtration problems, the

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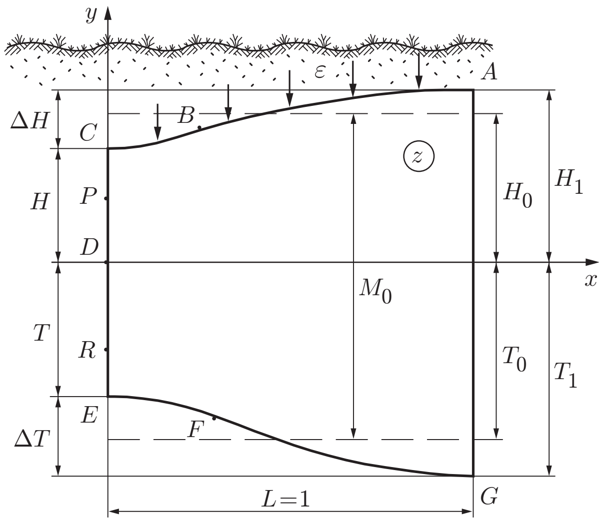

Fig. 1. Flow region in the fresh water fringe.

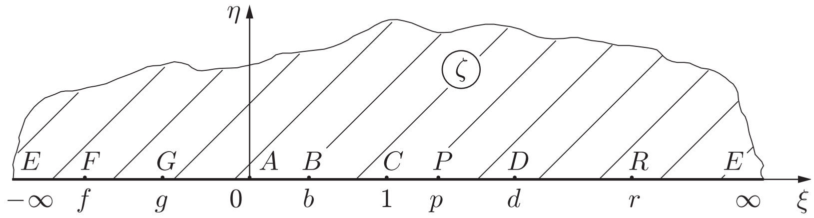

Fig. 2. Half-plane of the auxiliary complex variable.

velocity hodograph region $\bar{w} = w_x + i\, w_y$ is a circular polygon (Fig. 3), the function $w = w_x - i\, w_y$, which conformally maps the region of the complex filtration velocity w onto the half-plane $\operatorname{Im}\zeta \geqslant 0$ is the ratio of the two linearly independent solutions V_1 and V_2 of the following linear differential equation of the Fuchs class [2–4]:

$$\frac{d^2V}{d\zeta^2} + p(\zeta)\,\frac{dV}{d\zeta} + q(\zeta)V = 0. \tag{2}$$

Here

$$p(\zeta) = -\frac{1}{\zeta - f} + \frac{1}{2(\zeta - g)} + \frac{1}{2\zeta} - \frac{1}{\zeta - b} + \frac{1}{2(\zeta - 1)};$$

$$q(\zeta) = \frac{\zeta^3/2 + \mu_0\zeta^2 + \mu_1\zeta + \mu_2}{(\zeta - f)(\zeta - g)\zeta(\zeta - b)(\zeta - 1)}.$$

In addition to the mapping parameters b, f, and g to be determined, the coefficient $q(\zeta)$ of Eqs. (2) includes the so-called accessory (auxiliary) parameters μ_0, μ_1, and μ_2. They need not be determined if we use the Liouville formula

$$\frac{dw}{d\zeta} = \frac{d}{d\zeta}\Big(\frac{V_1}{V_2}\Big) = \frac{C\exp\left(-\int p(\zeta)d\zeta\right)}{V_2^2} = \frac{C(\zeta - f)(\zeta - b)}{V_2^2\sqrt{(\zeta - g)\zeta(\zeta - 1)}}. \tag{3}$$

Formula (3), in which C is a constant factor, includes only the coefficient $p(\zeta)$ and allows one to determine the functions V_1 and V_2 if the dependence $w(\zeta)$ is known.

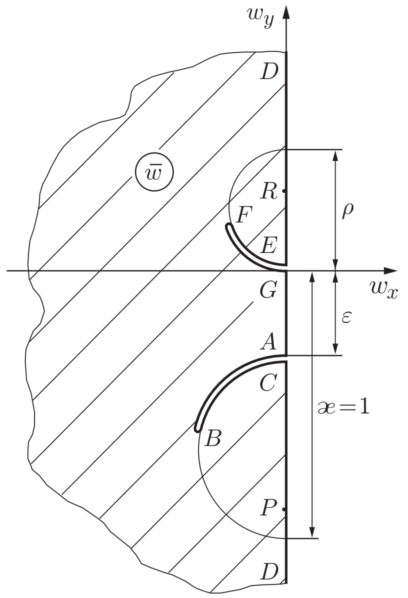

Fig. 3. Seepage velocity hodograph.

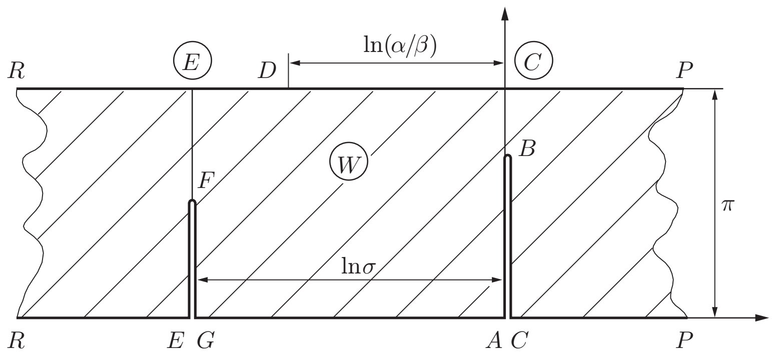

Fig. 4. Intermediate region in the map of the velocity hodograph onto the half-plane.

To find the latter dependence, we transform the velocity hodograph to a rectilinear polygon W (Fig. 4), which is then mapped onto the half-plane $\operatorname{Im}\zeta \geqslant 0$ (see Fig. 2). As a result, we have

$$W(\zeta) = \ln \frac{2i + \beta w}{2i\sigma - \alpha w} = i\, c_0 \int\limits_0^\zeta \Phi(u)\, du \qquad (c_0 > 0), \tag{4}$$

where

$$\Phi(u) = \frac{(b-u)(u-f)}{(p-u)(r-u)}\, \Phi_0(u), \qquad \Phi_0(u) = \frac{1}{\sqrt{(u-g)u(1-u)}},$$

$$\sigma = \frac{\sqrt{\epsilon+\rho} + \sqrt{\epsilon(1+\rho)}}{\sqrt{\epsilon+\rho} - \sqrt{\epsilon(1+\rho)}}, \qquad \alpha = \frac{\sigma-1}{\epsilon} + \sigma + 1, \qquad \beta = \frac{\sigma-1}{\epsilon} - \sigma - 1.$$

In Fig. 4, letters C and E in circles indicate the positions of the corresponding points in the limiting cases considered in Sec. 3.

We obtain the function $w(\zeta)$ by relation (4) and then the functions V_1 and V_2 from (3). At the same time, $w = d\omega/dz = \Omega/Z$, and, hence, the required functions Ω and Z can differ from the functions V_1 and V_2, respectively, by the same functional factor, whose form is established by analyzing its behavior in the neighborhood of each singular point. As a result, we arrive at the following relations [1]:

$$\frac{d\omega}{d\zeta} = -2c_1 \frac{F_1(\zeta)}{d-\zeta}\lambda(\zeta), \quad \frac{dz}{d\zeta} = ic_1 \frac{F_2(\zeta)}{d-\zeta}\lambda(\zeta), \quad \lambda(\zeta) = \sqrt{\frac{(\zeta-p)(\zeta-r)}{(\zeta-g)\zeta(\zeta-1)}}, \tag{5}$$

$$F_1(\zeta) = \sigma U - 1/U, \quad F_2(\zeta) = \alpha U + \beta/U, \quad U = \exp[W(\zeta)/2], \quad c_1 > 0.$$

For the function $\lambda(\zeta)$, we choose the branch that is positive at $\zeta > r$.

2. Determining the Mapping Parameters. Along with the constant factors c_0 and c_1, relations (4) and (5) contain the unknown affixes b, g, d, f, p, and r of the singular points. The primal formulation of the boundary-value problem adopted in the present study includes the determination of all indicated parameters.

As an internal problem we first consider the problem of determining the parameters c_0, b, d, f, and p, assuming that the quantities ϵ and ρ and the affixes g and r are specified. Using the known elements of the region W and taking into account the correspondence of the singular points in its mapping onto the half-plane $\operatorname{Im}\zeta \geqslant 0$ (see Figs. 2 and 4), we obtain the following system of equations based on (4):

$$\begin{gathered}
\int_0^1 \frac{(b-u)(u-f)\,du}{(p-u)(r-u)\sqrt{(u-g)u(1-u)}} = 0, \\
c_0 \int_g^0 \frac{(b-u)(u-f)\,du}{(p-u)(r-u)\sqrt{(u-g)(-u)(1-u)}} = \ln\sigma, \\
c_0 \int_1^d \frac{(b-u)(u-f)\,du}{(p-u)(r-u)\sqrt{(u-g)u(u-1)}} = \ln\frac{\alpha}{\beta}, \\
c_0(p-f)(p-b)\Phi_0(p) = r-p, \quad c_0(r-f)(r-b)\Phi_0(r) = r-p.
\end{gathered} \tag{6}$$

For convenience, we perform the replacement

$$k = \sqrt{\frac{-g}{1-g}}, \quad s = \sqrt{\frac{1-g}{r-g}}, \quad t = \sqrt{\frac{1-g}{p-g}}, \quad \Theta = \sqrt{\frac{1-g}{d-g}}, \tag{7}$$

under which specifying the mapping parameters g and r is equivalent to specifying the parameters k and s. The quantities t and Θ, related to the mapping parameters p and d, are to be determined alongside the quantities c_0, b, and f in system (6). This system is solved by a number of transformations using the elliptic integrals and functions presented in detail in [5].

Using the expansion

$$\frac{(b-u)(u-f)}{(p-u)(r-u)} = \frac{(r-f)(r-b)}{(r-p)(r-u)} - \frac{(p-f)(p-b)}{(r-p)(p-u)} - 1,$$

we write the first equation of system (6) in the form

$$\frac{\nu(r)-\nu(p)}{r-p} - K' = 0, \tag{8}$$

where

$$\nu(u) = \frac{(u-f)(u-f)}{u-1}\Pi\Big(-\frac{1}{u-1}, k'\Big); \quad K' = K(k'); \quad k' = \sqrt{1-k^2};$$

K and Π are complete elliptic integrals of the first and third kinds and the latter for $\gamma^2 < 0$ has the following representation:

$$\Pi(\gamma^2, k') = \frac{k'^2 K'}{k'^2-\gamma^2} - \frac{\pi\gamma^2\Lambda_0(\delta, k')}{2\sqrt{\gamma^2(1-\gamma^2)(\gamma^2-k'^2)}}, \qquad \delta = \arcsin\sqrt{\frac{\gamma^2}{\gamma^2-k'^2}}.$$

It contains the standardized lambda function

$$\Lambda_0(\delta, k') = (2/\pi)[E'F(\delta,k) + K'E(\delta,k) - K'F(\delta,k)],$$

by means of which the integral $\Pi(\gamma^2, k')$ is expressed in terms of the incomplete $F(\delta, k)$ and $E(\delta, k)$ and complete K' and E' elliptic integrals of the first and second kinds. In this case, Eq. (8) becomes

$$\frac{\nu_1(r)-\nu_1(p)}{r-p}+\frac{(r-f)(r-b)}{(r-p)(r-g)}\,K'-\frac{(p-f)(p-b)}{(r-p)(p-g)}\,K'=K',$$
$$\nu_1(u)=\frac{\pi(u-f)(u-b)\sqrt{1-g}}{2\sqrt{u(u-g)(u-1)}}\,\Lambda_0\Big(\arcsin\sqrt{\frac{1-g}{u-g}},k'\Big). \tag{9}$$

To save space, we further omit arcsin in all arguments of the incomplete elliptic integrals.

The parameters b and f are eliminated from Eq. (9) using the last two equations of system (6) with the function Φ_0 defined according to (4). Using formulas (7), we express one of the required quantities — the factor c_0 — in terms of the parameters k, s and t:

$$c_0k'=\pi/(2K')[\Lambda_0(s,k')-\Lambda_0(t,k')]+\Delta(s)/s-\Delta(t)/t, \tag{10}$$

where $\Delta(\chi)=\sqrt{(1-\chi^2)(1-k^2\chi^2)}$.

The second equation of system (6) is transformed similarly:

$$F(t,k)=F(s,k)+(\ln\sigma/\pi)K'. \tag{11}$$

From this, using the well-known relations of elliptic function theory, we obtain the following representation for the parameter t:

$$t=[s\,\Delta(\tau)+\tau\,\Delta(s)]/(1-k^2s^2\tau^2). \tag{12}$$

By virtue of a monotonic increase of the function $F(x,k)$ in the argument x Eq. (11) implies the inequalities

$$K/K'\geqslant\ln\sigma/\pi,\qquad F(s,k)\leqslant K-(\ln\sigma/\pi)K'.$$

They lead to the following constraints on the parameters k and s:

$$k_0\leqslant k<1,\qquad 0\leqslant s\leqslant s_0. \tag{13}$$

Here k_0 is the solution of the equation

$$K(k_0)/K(k_0')=\ln\sigma/\pi\qquad (k_0'=\sqrt{1-k_0^2}), \tag{14}$$

whose unique solvability is ensured, like the first constraint in (13), by a monotonic increase in the integral $K(k)$ in the modulus k. The value of s_0 is given by the equality

$$s_0=\operatorname{sn}\Big(K-\frac{\ln\sigma}{\pi}\,K',k\Big)=\sqrt{\frac{1-\tau^2}{1-k^2\tau^2}}\qquad \Big(\tau=\operatorname{sn}\Big(\frac{\ln\sigma}{\pi}\,K',k\Big)\Big). \tag{15}$$

The elliptic function sn is the inversion of an incomplete elliptic integral of the first kind.

The third equation of system (6) is reduced to the form

$$\frac{\pi}{2K'}\,[\Lambda_0(t,k')-\Lambda_0(s,k')][K-F(\Theta,k)]-K[Z(t,k)-Z(s,k)]$$
$$+\frac{\Delta(s)}{s}\,\Pi\Big(\Theta,\frac{1}{s^2},k\Big)-\frac{\Delta(t)}{t}\,\Pi\Big(\Theta,\frac{1}{t^2},k\Big)=\ln\sqrt{\frac{\alpha}{\beta}}. \tag{16}$$

In addition to the indicated elliptic integrals and functions, Eq. (16) includes the zeta function $Z(\delta,k)=E(\delta,k)-(E/K)F(\delta,k)$ and the incomplete elliptic integrals of the third kind $\Pi(\Theta,n,k)$, whose argument is related to the required mapping parameter d [see (7)]; $d\in(p,r)$ (see Fig. 2) and, hence, $\Theta\in(s,t)$. A monotonic decrease of the left side of Eq. (16) from ∞ to $-\infty$ as a function of the parameter Θ during its increase in the interval (s,t) is established analytically. This ensures unique solvability of Eq. (16) for Θ; the parameter d is thus uniquely determined.

The parameters b and f, which were earlier eliminated by means of the last two equations of system (6), are expressed in terms of those already obtained. The indicated equations are written as

$$p^2-(b+f)p+bf=a(r-p)P,\qquad r^2-(b+f)\,r+bf=a(r-p)R,$$

where

$$a = c_0^{-1}, \qquad P = \Phi_0^{-1}(p), \qquad R = \Phi_0^{-1}(r). \tag{17}$$

From this, we have

$$b + f = p + r - a(R - P), \qquad bf = pr - a(pR - rP),$$

and, hence, the required parameters should satisfy the quadratic equation

$$\Gamma(\gamma) = \gamma^2 + [p + r - a(R - P)]\gamma + pr - a(pR - rP) = 0. \tag{18}$$

Using relations (17), (7), and (10) and equalities (4) for $\Phi_0(r)$ and taking into account the behavior of the lambda function $\Lambda_0(\delta, k')$, we obtain the inequalities $\Gamma(g) < 0, \Gamma(0) < 0$, and $\Gamma(1) > 0$. In addition, $\lim\limits_{\gamma\to\pm\infty} \Gamma(\gamma) = \infty$, and, hence, Eq. (18) has two simple real roots that define the parameters $f \in (-\infty, g)$ and $b \in (0, 1)$.

Thus, if the parameters k and s and specified subject to constraints (13), the parameters c_0, p, d, b, and f are sequentially found from equalities (10) and (12) and Eqs. (16) and (18); this is done using the transient relations (7).

The expression for the coefficient c_1 in relations (5) can be obtained from the first of these relations written for the segment PD (see Fig. 4). For $p < \zeta < d$, from (4) we have

$$W(\zeta) = W_1 - \ln(\alpha/\beta) + i\pi, \tag{19}$$

where

$$W_1 = c_0 \int\limits_{\zeta}^{d} \frac{(u - b)(u - f)\, du}{(u - p)(r - u)\sqrt{(u - g)u(u - 1)}},$$

and the first equation in (5) becomes

$$\frac{d\omega}{d\zeta} = \frac{2c_1 F_1(\zeta)|\lambda(\zeta)|}{d - \zeta},$$

where

$$F_1(\zeta) = \frac{\beta\sigma U_1 + 1/U_1}{\sqrt{\alpha\beta}}; \qquad U_1 = \exp\Big[\frac{W_1(\zeta)}{2}\Big]; \qquad |\lambda(\zeta)| = \sqrt{\frac{(\zeta - p)(r - \zeta)}{(\zeta - g)\zeta(\zeta - 1)}}.$$

From this, using the second condition of (1) on the segment CD, we obtain

$$\omega(\zeta) = 2c_1 \int\limits_{p}^{\zeta} \frac{F_1(\zeta)|\lambda(\zeta)|\, d\zeta}{d - \zeta} + \varphi(p) \qquad (p < \zeta < d). \tag{20}$$

According to (1), in passage to the segment ED the left side of equalities (20) changes by the value $i\epsilon$ of the total flow of infiltration water within the seepage region. Equating this value to the increment of the right side of equalities (20) $2c_1 i\pi F_1(d)|\lambda(d)|$, we obtain

$$c_1 = \frac{\epsilon\sqrt{\alpha\beta}}{2\pi(\beta\sigma + \alpha)} \sqrt{\frac{(d - g)d(d - 1)}{(d - p)(r - d)}}. \tag{21}$$

To find the parameters g and r, we use the relations

$$-\int\limits_{0}^{1} y(\zeta)\, \frac{dx(\zeta)}{d\zeta}\, d\zeta = H_0, \qquad -\int\limits_{-\infty}^{g} y(\zeta)\, \frac{dx(\zeta)}{d\zeta}\, d\zeta = T_0, \tag{22}$$

in which the left sides are the fresh water volumes above and below the drains in the fringe formed and the right sides are those before the actuation of the drains. Equalities (22) are based on the assumption that these volumes remain constant during the formation of the fringe.

The first equation of system (22) contains the coordinates of the points of the depression curve, whose parametric equations are obtained from the second relation of (5) written for the segment AC $(0 \leqslant \zeta \leqslant 1)$:

$$x(\zeta) = c_1(\alpha+\beta)\int\limits_{\zeta}^{1} \frac{\cos V_1(v)\,|\lambda(v)|}{d-v}\,dv, \quad y(\zeta) = H + c_1(\alpha-\beta)\int\limits_{\zeta}^{1} \frac{\sin V_1(v)\,|\lambda(v)|}{d-v}\,dv,$$

$$V_1(v) = \frac{c_0}{2}\int\limits_0^v \Phi(u)\,du. \tag{23}$$

For the function $\Phi(u)$, the representation (4) holds true.

According to (23), the first equations of (22) can be written as

$$H + c_1^2(\alpha^2-\beta^2)\int\limits_0^1 \Big[\int\limits_{\zeta}^{1} \frac{\sin V_1(v)|\lambda(v)|}{d-v}\,dv\Big]\frac{\cos V_1(\zeta)\,|\lambda(\zeta)|}{d-\zeta}\,d\zeta = H_0. \tag{24}$$

The left side of this equation includes the length H of the boundary segment CD, which consists of two segments separated by the point P. Taking into account the nature of change in the functions $F_2(\zeta)$ and $\lambda(\zeta)$ in passage through the indicated point and integrating the second equation of (5), we obtain

$$H = c_1\int\limits_1^p \frac{\alpha U + \beta U^{-1}}{d-\zeta}\,|\lambda(\zeta)|\,d\zeta + 2c_1\sqrt{\alpha\beta}\int\limits_p^d \frac{\sinh[W_1(\zeta)/2]}{d-\zeta}\,|\lambda(\zeta)|\,d\zeta, \tag{25}$$

where

$$U = \exp\frac{W(\zeta)}{2}; \qquad W(\zeta) = c_0\int\limits_1^{\zeta} \frac{(u-b)(u-f)\,du}{(p-u)(r-u)\sqrt{(u-g)u(1-u)}}.$$

The function $W_1(\zeta)$ is defined in (19).

Next, transforming relations (4) and (5) on the interface EG $(-\infty < \zeta \leqslant g)$, we arrive at its parametric equations

$$x(\zeta) = \frac{c_1(\beta\sigma+\alpha)}{\sqrt{\sigma}}\int\limits_{-\infty}^{\zeta} \frac{\cos V_2(v)|\lambda(v)|}{d-v}\,dv, \qquad y(\zeta) = -T - \frac{c_1(\beta\sigma-\alpha)}{\sqrt{\sigma}}\int\limits_{-\infty}^{\zeta} \frac{\sin V_2(v)|\lambda(v)|}{d-v}\,dv,$$

$$V_2(v) = \frac{c_0}{2}\int\limits_{-\infty}^{v} \frac{(b-u)(f-u)\,du}{(p-u)(r-u)\sqrt{(g-u)(-u)(1-u)}},$$

and, as a result, the second equation of (22) is written as

$$T + \frac{c_1^2(\beta^2\sigma^2-\alpha^2)}{\sigma}\int\limits_{\infty}^{g}\Big[\int\limits_{\infty}^{\zeta} \frac{\sin V_2(v)\,|\lambda(v)|}{d-v}\,dv\Big]\frac{\cos V_2(\zeta)\,|\lambda(\zeta)|}{d-\zeta}\,d\zeta = T_0. \tag{26}$$

From this relation, using (4) and (5), we obtain the following expression for the quantity T:

$$T = 2c_1\sqrt{\alpha\beta}\int\limits_d^r \frac{\sinh[W_1(\zeta)/2]}{\zeta-d}\sqrt{\frac{(\zeta-p)(r-\zeta)}{(\zeta-g)\zeta(\zeta-1)}}\,d\zeta$$

$$+\frac{c_1}{\sqrt{\sigma}}\int\limits_r^{\infty} \frac{\beta\sigma U + \alpha U^{-1}}{\zeta-d}\sqrt{\frac{(\zeta-p)(\zeta-r)}{(\zeta-g)\zeta(\zeta-1)}}\,d\zeta, \tag{27}$$

where

$$U = \exp\frac{W_2(\zeta)}{2}; \qquad W_2(\zeta) = c_0\int\limits_{\zeta}^{\infty} \frac{(u-b)(u-f)\,du}{(u-p)(u-r)\sqrt{(u-g)u(u-1)}}.$$

Thus, the problem reduces to finding the parameters k and s for specified quantities ϵ, ρ, and M_0 and one of the quantities H_0 or T_0.

3. Limiting Drainage Regimes. Let us elucidate the physical meaning of constraints (13) on the values of the parameter s.

Case $s = s_0$. From relations (12) and (15) and formula (10) for the function Δ it follows that $t = 1$, and by virtue of the third equality of (7), we have $p = 1$. Next, from the last two equations of system (6) taking into account (4), we have

$$1 - b = (r - f)(r - b)/(1 - f)\Phi_0(r)/\Phi_0(1) = 0, \qquad b = 1,$$

and, thus, the inflection point B of the depression curve and the point P of the segment CD are made coincident with the point C. As a result, the half-circle $|\bar{w} + i(1 + \epsilon)/2| < (1 - \epsilon)/2$, $\operatorname{Re}\bar{w} \leqslant 0$ vanishes from the hodograph and the half-band $\operatorname{Re} W > 0$, $0 \leqslant \operatorname{Im} W \leqslant \pi$ vanishes from the region W (see Figs. 3 and 4). For the point C, at which $\bar{w} = -i$, we have $W = i\pi$, according to (4), and next, from Eqs. (23), we obtain

$$\frac{dy}{dx} = \frac{\alpha - \beta}{\alpha + \beta} \tan V_1(1) = \frac{\alpha - \beta}{\alpha + \beta} \tan \frac{|W(1)|}{2} = \infty,$$

i.e., the point C becomes a cusp.

Let us consider the segment CD, on which $\bar{w} = i\,w_y$ and $w_y = d\varphi/dy \leqslant -1$. From the last inequality and the relation

$$p/(\rho_1 g) = -\varphi - y, \tag{28}$$

which links the reduced potential φ of the filtration velocity to the pressure p, it follows that $dp/dy = -\rho_1 g(w_y + 1) \leqslant 0$; the equality holds only at the point C. On the remaining segment CD, the pressure is below atmospheric one, and its further arbitrary small reduction should result in air breakthrough into the drain. Vedernikov [6] was the first to point out the feasibility and physical reasons of such a critical regime.

Case $s = 0$. From the second relation in (7) and the fourth equation of (6) it follows that in this case, $r = -f = \infty$: the points R and F coincide with the point E, which becomes a cusp of the interface; at this point, $\bar{w} = i\rho$. As a result, the half-circle $|\bar{w} - i\rho/2| < \rho/2$, $\operatorname{Re}\bar{w} \leqslant 0$ vanishes from the hodograph, and the half-circle $\operatorname{Re} W < \ln\sigma$, $0 \leqslant \operatorname{Im} W \leqslant \pi$ vanishes from the region W (see Figs. 2 and 3). On the segment ED, where $\bar{w} = iw_y$, we have $w_y \geqslant \rho$, and using Eq. (28), we obtain $dp/dy = -\rho_1 g(w_y + 1) \leqslant -\rho_2 g$. In this case, the equality describing the state of hydrostatic equilibrium in the saline water zone is valid only at the point E, on which, figuratively speaking, the dynamic equilibrium in the fresh water flow reposes in the case considered. On the remaining segment CE, the hydrodynamic pressure gradient exceeds the stabilizing effect of gravity on the saline water, and a further arbitrarily small pressure reduction should involve them in the flow. This regime was first detected in [7].

Thus, in each of the indicated limiting cases, the drain flow rate is maximally admissible and, hence, should be determined first of all. However, it is not known beforehand what critical situation arises as the drain rate increases for a particular drain depth. Below we consider a double critical flow regime in the fringe, which plays a key role in the solution of this problem. This regime combines the indicated regimes and the regime described for the first time in an analysis [8] of a similar filtration model.

Case $s_0 = 0$. In this limiting regime, both half-circles vanish from the hodograph and the region W becomes a rectangle mapped onto the half-plane $\operatorname{Im}\zeta \geqslant 0$ by means of the relation

$$W(\zeta) = i\,\frac{\pi}{K_0'}\,F\Big(\sqrt{\frac{\zeta}{k_0^2 + k_0'^2\zeta^2}}, k_0'\Big), \tag{29}$$

where $K_0' = K(k_0')$ and $k_0' = \sqrt{1 - k^2}$. The elliptic integral modules k_0 and k_0' are determined from Eq. (14), which is identical to Eq. (11) in the case considered.

Equations (16) leads to

$$\Theta = \operatorname{sn}\left((K_0'/\pi)\ln(\beta\sigma/\alpha), k_0\right),$$

and equalities (21) become

$$c_1 = \epsilon\sqrt{\alpha\beta d(d - g)}/(2\pi(\beta\sigma + \alpha)). \tag{30}$$

As a result, with specification of the quantities ϵ and ρ, the parameters d and g and the factor c_1 are determined uniquely. Next, all the main (geometrical characteristics pertaining to the fringe, including the thickness M_0 of the fresh water layer, are evaluated. In the case, equalities (24) and (26), which define the quantities H_0 and T_0, include the relations

$$V_1(v) = \frac{|W(v)|}{2}, \qquad V_2(v) = \frac{\pi}{2K_0'} F\Big(\sqrt{1 - \frac{g}{\zeta}}, k_0'\Big), \qquad \lambda(\zeta) = \frac{1}{\sqrt{\zeta(\zeta - g)}},$$

$$H = c_1 \int\limits_1^d \frac{(\alpha U^{-1} - \beta U)\, d\zeta}{(d - \zeta)\sqrt{\zeta(\zeta - g)}}, \qquad T = c_1 \int\limits_d^\infty \frac{(\beta U - \alpha U^{-1})\, d\zeta}{(\zeta - d)\sqrt{\zeta(\zeta - g)}}, \tag{31}$$

$$U = \exp\Big[\frac{\pi}{2K_0'} F\Big(\sqrt{1 - \frac{1}{\zeta}}, k_0'\Big)\Big].$$

The function $W(v)$ is defined in (29).

4. Numerical Calculations and Flow Analysis. The computational algorithms implemented for the boundary-value problem considered are based on a numerical solution of the transcendental equations for the unknown parameters in the relations used to find the characteristics of the filtration process. The unique solvability of equations of the form $F(\mu) = 0$ is ensured by the monotonicity of the function $F(\mu)$ and the difference between its signs at the ends of the range of the required parameter μ. For all equations considered below and used to find the parameters k and s and (for the limiting drainage regimes) the parameter ϵ, such behavior of the functions appearing in these equation is previously established numerically. The algorithm of determining the remaining unknown parameters described in Sec. 2 was validated analytically.

At the first stage, the value ϵ_{**} of the parameter ϵ corresponding to the double critical regime is found from the equation

$$F(\epsilon) = f[\epsilon, k_0(\epsilon)] = H_0 + T_0 = M_0 \tag{32}$$

with a specified right side M_0.

The function $f[\epsilon, k_0(\epsilon)]$, in which the dependence $k_0(\epsilon)$ is defined by Eq. (14) and the function by equalities (24), (26), (30), and (31), increases monotonically from 0 to ∞ as the parameter ϵ increases from 0 to 1. We note that according to (4) and (14), with such an increase, the parameter σ increases in the interval $(0, \infty)$, and the modulus k_0 in the interval (0, 1). By virtue of the indicated behavior of the function $f[\epsilon, k_0(\epsilon)]$, the quantity ϵ_{**} — the maximum possible drain rates — is uniquely calculated from Eq. (32) (for specified values of the physical parameters M_0 and ρ). During the solution of Eq. (32), we find the depth $H_* = H_0(\epsilon_{**})$ of drain location in the fresh water layer at which both moving boundaries of the fringe appear on the verge of destabilization after the quantity ϵ reaches the value ϵ_{**}.

Depending on the relation between the quantities H_* and H_0, it is established which of the two above-mentioned simple critical regimes occurs with increase in the flow rate of the drain located at depth H_0. The maximum attainable drain flow rate $\epsilon_* \in (0, \epsilon_{**})$ is determined along with the parameter k from the equations

$$F_1(\epsilon) = f_1[\epsilon, k(\epsilon)] = H_0 \quad (\epsilon \in (0, \epsilon_{**})), \qquad F_2(k) = f_2(\epsilon, k) = M_0 \quad (k \in (k_0, 1)). \tag{33}$$

The functions f_1 and f_2 are defined by equalities (24)–(27), which should be modified as applied to the critical regime considered.

System (33) is solved using an iterative procedure. The first equation of (33), whose left side is a complex function of the parameter ϵ is solved in the external cycle. The lower boundary $k_0(\epsilon)$ of the admissible values of the parameter k is obtained from relation (14) for each value of the parameter $\epsilon \in (0, \epsilon_{**})$. The parameter k is found using the second equation of (33). The left side of this equation is defined by expressions (22) for the quantities H_0 and T_0 transformed for the corresponding limiting case, and increases from a certain value m_0 to ∞ as k increases from k_0 to 1. The quantities m_0 and M_0 are the thicknesses of the fresh water layer in the double critical regime for a chosen value of the parameter ϵ and for its maximum possible value ϵ_{**}, respectively. Taking into account the nature of the dependence $M_0(\epsilon)$ observed for this regime, we conclude that $m_0 < M_0$, and, hence, the parameter k is uniquely calculated from the second equation of (33).

As the parameter ϵ increases from 0 to ϵ_{**}, the function $F_1(\epsilon)$ increases from 0 to H_* within the framework of the critical regime related to the depression curve, and in the second limiting case, it decreases from M_0 to H_*. Thus, the first equation of (33) defines the drain flow rate in the corresponding simple critical regime that is maximally admissible for a specified depth H_0 of drain locations. For $M_0 = 2$ and $\rho = 0.02$, the indicated relation is characterized by the parameter values $\epsilon_* = 0.0216$, 0.0554, 0.2597, 0.4944, **0.5509**, 0.4718, 0.1303, 0.0200, and 0.0012 computed for $H_0 = 0.05$, 0.1, 0.3, 0.5, **0.5468**, 0.6, 1.0, 1.5, and 1.9, respectively (the bold figures refer to the double critical regime).

Next, we consider the general case assuming that $\epsilon \in (0, \epsilon_*)$. In this case, the flow to the drains at depth H_0 occurs in the normal drainage regime. The calculations at this stage are based on the relations presented in Secs. 1 and 2 and reduce to finding the parameters k and s from the system

$$G_1(k) = g_1[k, s(k)] = M_0, \qquad G_2(s) = g_2(k, s) = H_0. \tag{34}$$

The functions g_1 and g_2 are defined by equalities (24)–(27).

System (34) is also solved by a two-stage iterative procedure. In its external cycle, the parameter $k \in (k_0,\ 1)$ is found from the first equation of the system. The dependence $s = s(k)$ appearing in it is defined by the second equation of (34), which is solved in the internal cycle. For any value of $k \in (k_0, 1)$, the function g_2 decreases monotonically in the parameter $s \in (0, s_0(k))$ and takes values in the interval (H_{01}, H_{02}). The ends of this interval are the values of H_0 in the two simple critical drainage regimes for a given value of ϵ, whereas the fixed value of H_0 on the right side of the second equation of (34) corresponds to the critical regime with $\epsilon_* > \epsilon$. From this, taking into account the above-mentioned relation between the quantities H_0 and ϵ in the critical regimes, we obtain the inequalities $H_{01} < H_0 < H_{02}$, which ensure unique solvability of the second equation of system (34).

The left side $G_1(k)$ of the first equation of the system in question increases from M_{00} for $k = k_0$ to ∞ for $k = 1$. The quantities M_{00} and M_0 are the thicknesses of the fresh water layer in the double critical regime for a specified value of the parameter ϵ and for its maximum possible value ϵ_{**} achievable in the indicated limiting case, respectively. In view of this and taking into account the nature of the dependence $f[\epsilon, k_0(\epsilon)]$ established in the consideration of Eq. (32), we conclude that the specified value of M_0 satisfies the inequality $M_{00} < M_0$, and, hence, the parameter k is uniquely determined from the first equation of system (34).

In the solution of the problem in the direct statement, the most labor-consuming stage of the computational procedure described above is finding the unknown mapping parameters. The associated repeated numerical solution of complex transcendental equations is implemented by a standard iterative procedure which includes isolation of a reduced interval containing the required parameter with subsequent interpolations.

Elliptic integrals and functions are calculated using the cost-effective algorithms described in [9]. A special approach is required for the improper integrals included in the computational formulas, each of which need to be reduced to computational form. They are dominated by integrals of the form $\int\limits_a^b f(u)\,du/\sqrt{(u-a)(b-u)}$, in which the function $f(u)$ is continuous. The singularities of the integrand function at both ends of the integration interval are eliminated by the replacement $u = a + (b-a)v^2(2-v^2)$, which yields

$$u - a = (b-a)v^2(2-v^2), \quad b - u = (b-a)(1-v^2)^2, \quad du = 4(b-a)v(1-v^2)\,dv.$$

In each integral over a infinite interval $[a, \infty)$, the replacement $u = a/(v^2(2-v^2))$ is implemented.

The transformed integrals are evaluated by the Simpson formula in the interval $[0, 1]$, which is unified for all integrals. The procedure is accelerated by distinguishing segments of the interval on which the integrand function changes relatively rapidly; these segments adjoin, as a rule, the ends of the main interval of integration.

To illustrate the computational procedure described above, we give the results of calculations for $M_0 = 2$, $H_0 = 0.3$, and $\rho = 0.02$. In the primal formulation, the seepage rate ϵ must also be specified but, according to the preliminary analysis, it is previously necessary to establish the range of admissible values of this quantity.

For the indicated input parameters in the first stage of calculations for the double critical regime, we found the maximum possible drain rate $\epsilon_{**} = 0.5509$ and the drain depth $H_* = 0.5468$ at which this maximum is reached. For all other values $\epsilon \in (0, \epsilon_{**})$, the drains operate at the depth H_* in the normal drainage regime.

For the chosen value $H_0 = 0.3$, the maximum admissible value $\epsilon_* = 0.2596$ of the parameter ϵ is reached in the critical regime related to the depression curve. The amplitude ΔH of the ordinate of the points of the depression curve (see Fig. 1) decreases from 0.4656 in the double critical regime to 0.3010 in the regime considered, remaining considerable, whereas the value of ΔT decreases from 0.6260 to 0.0945, i.e., at higher locations of the drains, their effect on the saline water is weakened. The drain rate has the greatest effect on the minimum thickness M of the fresh water layer: for the version considered, $M = 1.1796$ in the most intense double critical regime, and for $H_0 = 0.3$, we calculated $M = 1.7151$ for the simple critical regime (for $\epsilon = \epsilon_* = 0.2596$) and $M = 1.9434$ for the normal drainage regime (at $\epsilon = 0.1$).

In addition to the parameter ϵ, which is of significance as one of the controlled physical factors governing the fringe flow, the saline water density ρ_2 appearing in the parameter ρ also plays an important role. We note that for real salinity of groundwater, small (hundredths) values of this parameter are of primary interest for hydromelioration.

From the first condition of (1) on the interface EG, we obtain the equality

$$\Delta T = y_E - y_G = (\varphi_E - \varphi_G)/\rho, \tag{35}$$

which reflects an increase in the sensitivity of saline water to drainage with reduction in water density. According to (4), as $\rho \to 0$, we have

$$\sigma = \frac{[\sqrt{\rho + \epsilon} + \sqrt{\epsilon(1 + \rho)}]^2}{\rho(1 - \epsilon)} \approx \frac{4\epsilon^2}{\rho(1 - \epsilon)}. \tag{36}$$

From (36) and (14) it follows that as $\rho \to 0$, the parameter σ increases without bound, together with the lower boundary k_0 of the admissible values of the parameter k, which should accompany deepening of the interface. This regularity, supported by calculations, can be substantiated theoretically: it is established that $\lim\limits_{\rho \to 0} T = \infty$. In the limit (for $\rho = 0$), we obtain the model of fresh water drainage in a bed of unlimited thickness considered in [6].

As follows from the asymptotic representation (36), a reduction in ρ is compensated by a decrease in the parameter ϵ. This implies that a decrease in the rate of fresh water drainage in the fringe allows weakly saline groundwater to be kept immovable for a fixed thickness of the fresh water layer, but in this case, according to calculations, the drains recede from the interface. Thus, for $M_0 = 2$ and $\rho = 10^{-4}$ for the double critical regime, we calculated $\epsilon = 0.0080$ and $H_0 = 0.0230$.

Let us consider the second limiting case: $\rho \to \infty$. In this case, the saline groundwater "solidify" and there is a scheme of drainage of infiltration water in the soil layer with the interface becoming a horizontal impermeable bed [according to equalities (35)]. In [10], this important hydromelioration problem was solved approximately by replacing the free surface of the flow by a fixed horizontal boundary. In the primal formulation, the degree of complexity of the boundary-value problem for the model with an impermeable bed is the same as for the fringe flow. In this case, however, the calculations are reduced and the flow analysis is simplified because in the model with an impermeable bed, the drainage rate is regulated only by the critical regime for the free surface.

In the problem considered, it is possible to approach the model with an impermeable bed by increasing the parameter ρ. Thus, for $M_0 = 2$ and $\rho = 10^6$ even in the double critical regime, where the ordinates of the free boundaries change in wide ranges, we have $\Delta T = 8.7 \cdot 10^{-6}$ (the interface becomes a horizontal impermeable bed; in this case, $\epsilon_* = 0.9878$ and $T = 3.3 \cdot 10^{-7}$). This approach of the sink to the interface is due to the fact that in the double critical regime, the drainage rate is maximal and in the limiting case $\rho \to \infty$, this maximum should be reached for drain location at the impermeable bed, where it is the most distant from the depression curve. For the above-stated value of the parameter ρ, only this moving boundary is susceptible to destabilization at any depth of drain location in the fresh water layer.

According to calculations, variation in the parameter ρ for fixed values of ϵ, M_0, and H_0 results in deformation of the interface and, at the same time, has almost no effect on the position of the depression curve.

5. Some Modifications of the Problem Formulation. The examined model for the flow in the fresh water fringe is based on the assumption of conservation of dynamic equilibrium of the flow with air and with the saline ground water, which should be kept immovable. The first of the indicated conditions is immutable since the examined hydrodynamic model ceases to operate with breakthrough of air into the drain. If drainage intensification leads to involvement of saline water in the flow, the saline water is partly displaced by fresh water and in the limit, a stationary flow of fresh water above the undisplaced saline water is formed. This flow is still described by the boundary-value problem considered and proceeds in the critical regime on the interface. If the saline water is isolated from external sources and sinks, the mean depth of this interface will be below the initial depth by a value determined by the volume of drained saline water.

With such a physical basis, following [11], it is possible to formulate the problem of calculating the thickness H_0 of the fresh water layer in a soil that initially contained only saline groundwater. One of the criteria of an optimal hydromeliorative regime of soils is to ensure groundwater depth that eliminates its intense vaporization. In view of this, in the calculations, we specify the mean height H_0 of the free surface above the drain water line and normalize all geometrical parameters of the fringe to this quantity. All calculations are performed in the double critical regime, in which the extreme values of the fringe flow characteristics are reached. For this regime, we need to make a single correction in the computational algorithm described above (see Sec. **4**), using now the quantity H_0 to find the parameter $\epsilon = \epsilon_{**}$. In the calculations, we still set $L = 1$, and after completion of the calculations, all the linear values obtained should be normalized by H_0.

For $\rho = 0.02$ and $L/H_0 = 0.5$, 2, 10, and 100, computations using the above scheme yield $M_0 = 1.8131$, 3.8538, 9.8108, and 27.0804 and $\epsilon_{**} = 0.9939$, 0.5003, 0.0559, and 0.0024, respectively. From this, it is concluded that the deeper displacement of saline water by fresh water can be achieved by spacing drains more widely apart but this considerably reduces the drainage capabilities limited by the free surface of the fringe. The maximum attainable depth of the desalination zone is significantly affected by the saline water density. Thus, for $\rho = 0.1$ and $L/H_0 = 10$, the value $M_0 = 5.5899$ was computed, although the value $\epsilon_{**} = 0.0533$ is close to that obtained for $\rho = 0.02$ and $L/H_0 = 10$. Finally, the groundwater desalination depth is larger the deeper the drain location. It should be borne in mind that in such estimates of the end results of displacement of saline water by fresh water, the question of the dynamics of this process remains open; it can be investigated only on the basis of the corresponding nonstationary boundary-value problem.

For applications of mathematical drainage models, the question of drainage schematization is a fundamental one. The above formulation of the problem is based on specification of the drain flow rate, whose maintenance at a certain level requires the corresponding pressure reduction in the drain pipes. Fresh groundwater intakes and so-called vacuum drainage melioration systems operate in this regime [12]. In most cases, however, the rate of groundwater drainage on irrigated land is determined by the head at the interface between the drains and the soil, and the latter factor is determined by the drain hole pressure and the hydraulic resistance of the drain filters. In this situation, it is reasonable to specify the pressure on the external contour of the drain. In this case, the attainment of the drain flow rate compensating for infiltration of certain rate ϵ can be preceded by a certain (theoretically infinite) period of groundwater rise to a level at which the increased operating head overcomes the flow resistance at the entrance to the drains. However, as noted above, this rise should not exceed the value admitted under hydromeliorative criteria, which can be satisfied in this case by choosing appropriate depth and frequency of drain location. In view of this, the number of specified values includes the maximum height H_1 of the free surface of groundwater above the drain line (see Fig. 1).

The formulation of the problem rules out the critical regime on the depression curve since a prerequisite for this regime is rarefaction of the flow by drains. As regards the second limiting drainage regime ($s = 0$), it is used as the basis in numerical calculations in this case. The main stage in the calculations is finding the elliptic integral modulus k from the equation resulting from specification of the pressure p_1 at a certain point D_1 belonging to the segment AD and located on the drain contour. The expression for p_1 is obtained using relation (28) and the dependence $\varphi(d_1)$ based on representation (5) for the function $\omega(\zeta)$. The affix d_1 of the point D_1 is determined previously from the equation $z(d_1) = iy_{D_1}$. All computed linear flow characteristics are normalized by the quantity H_1.

For $\rho = 0.02$, $y_{D_1} = 0.01$, $p_1 = 0.5$ and $\epsilon = 0.1$, 0.01, and 0.001, we obtained $L = 2.4662$, 16.3451, and 85.4195 and $T_0 = 2.3305$, 7.5835, and 14.1370. From this it follows that as the infiltration rate decreases, the required hydromeliorative regime is ensured by more widely spaced drains with a simultaneous increase in the depth of the soil desalination zone. In addition, soil drainage is promoted by drain deepening and enlargement and by reducing the pressure on the drain contour. Thus, for $p_1 = 0.2$, the above-stated values of the input parameters ρ and y_{D_1}, and for $\epsilon = 0.01$, we obtained $L = 23.9323$ and $T_0 = 11.4781$. As was noted at the beginning of Sec. 5, in both modifications of the initial formulation of the boundary-value problem, the saline groundwater level does not play an important role before actuation of the drains.

Conclusions. A mathematical model for horizontal pipe drainage in a fresh infiltration water fringe above saline groundwater was developed and studied. The model is based on a multiparameter boundary-value problem solved in primal formulation using methods of the analytical theory of linear differential equations; emphasis was placed on finding the unknown mapping parameters. The flow was analyzed using an approach (tested earlier [8] for similar problem) that involves detection of the limiting drainage regimes determining the boundaries of the examined filtration process within the framework of the boundary-value problem. The problem, which is oriented primarily toward calculating fresh water intakes, can also be useful in solving some problems of hydromelioration of irrigated lands.

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